Electromagnetic field in Schwarzschild black hole background. Analytical treatment and numerical simulation

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Abstract.
The problem of description of electromagnetic field in the Schwarzschild black hole background can be reduced to an ordinary differential equation of the Heun type. Possible solutions are listed in terms of confluent Heun functions. We study an equivalent Schrödinger like radial equation with an effective barrier potential, generated by Schwarzschild gravitational field. Exact condition insuring existence of two turning points and thereby possibility to have quantum mechanical tunneling effect are found analytically and examined numerically.

1 Introduction. Setting the problem

The problem of description of electromagnetic field in the Schwarzschild black hole background can be reduced to an ordinary differential equation of the form [1][1]

$$\frac{d^2G}{dr^2} + \frac{M}{r(r-r_0)}\,\frac{dG}{dr} + \left(\frac{k^2 r^2}{(r-r_0)^2} - \frac{j(j+1)}{r(r-r_0)} \right) G = 0\ . \tag{1}$$

r_0 is a Schwarzschild radius determined by the mass of the black hole [2]:
Little asteroid: 10^{18}kg , $r_0 = 10^{-9}$ meter
The Earth: $5,97 \cdot 10^{24}$ kg, $r_0 = 10^{-2}$ meter
The Sun: $1,99 \cdot 10^{30}$ kg, $r_0 = 3 \cdot 10^{3}$ meter
Massive star: $2 \cdot 10^{31}$ kg, $r_0 = 3 \cdot 10^{4}$ meter

In the dimensionless variable $x = r/r_0$, the last equations reads

$$\frac{d^2G}{dx^2} + \left(\frac{1}{x-1} - \frac{1}{x} \right) \frac{dG}{dx} + \left(r_0^2 k^2 + \frac{j(j+1)}{x} + \frac{2r_0^2k^2 - j(j+1)}{x-1} + \frac{r_0^2 k^2}{(x-1)^2} \right) G = 0. \tag{2}$$

[1]In fact, the same type of equation arises when treating a scalar massive particle – difference consists only in formal change $k^2 = \epsilon^2/\hbar^2 c^2 - m^2c^2/\hbar^2$, where ϵ stands for energy of the particle and m represents its mass. Physical dimension of k is meter^{-1}.

Let us introduce dimensionless parameter $p = r_0 k = r_0 \frac{mc}{\hbar}\sqrt{(\frac{\epsilon}{mc^2})^2 - 1} = \frac{r_0}{\lambda}\sqrt{\mu^2-1}$. For electron $\lambda = 2,4\cdot 10^{-12}$ meter, therefore for a massive star, parameter p is $p \sim 10^{15}\sqrt{\mu^2-1}$. At very small energy, $\mu = 1+\delta$, we have $\sqrt{\mu^2-1} = \sqrt{1+2\delta+\delta^2-1} = \sqrt{2\delta} \sim \frac{v}{c}$, v is velocity of a slowly moving particle. Therefore, parameter p varies according to $p \in (0, +\infty)$. Then equation (2) reads

$$\frac{d^2G}{dx^2} + \left(\frac{1}{x-1} - \frac{1}{x}\right)\frac{dG}{dx} + \left(p^2 + \frac{j(j+1)}{x} + \frac{2p^2 - j(j+1)}{x-1} + \frac{p^2}{(x-1)^2}\right) G = 0. \quad (3)$$

It is easily to find possible behavior of solutions near singular points:

$$x \to 1, G \sim (x-1)^{\alpha}, \alpha = \pm ip;$$
$$x \to \infty,\ G \sim e^{\gamma x},\ \gamma = \pm ip; \qquad x \to 0,\ G \sim x^{\beta},\ \beta = 0 \text{ or } 2. \quad (4)$$

Physical region for variable x is given by the interval $x \in (1, +\infty)$. It is convenient to use the following parameter $p = Mk$; below it will be named as a momentum (at infinity $x \to$) depending upon the sign $\pm$ one says: the (spherical) wave goes to infinity (e^{-ipx}) or from infinity (e^{-ipx}). Physical behavior of solutions in the region $x = 1+0$ can be understood better if one introduces a special new variable

$$\ln(x-1) = y, \qquad x = 1 + e^y , \qquad x \to 1+0 \iff \qquad y \to -\infty , \quad (5)$$
$$x \to +\infty \qquad \iff \qquad y \to +\infty , \qquad x = 2 \qquad \iff \qquad y = 0$$

then

$$(x-1)^{\pm ip} = e^{\pm ip \ln(x-1)} = e^{\pm ipy} . \quad (6)$$

In conventional physical terminology, they can be considered as spherical waves, spreading from the left-hand side ($y \to -\infty$) (from or to infinity $y \to -\infty$ depending on the sign $\mp$). With the use of substitution $G = (x-1)^{\alpha}\, x^{\beta} e^{\gamma x}\, g(x)$, equation (3) leads to

$$\frac{d^2g}{dx^2} + \left(2\gamma - \frac{1-2\beta}{x} + \frac{1+2\alpha}{x-1}\right)\frac{dg}{dx} + \left(p^2 + \gamma^2 + \frac{p^2+\alpha^2}{(x-1)^2} + \frac{\beta(\beta-2)}{x^2} + \right.$$
$$\left. + \frac{j(j+1)+\alpha-\beta-\gamma-2\alpha\beta+2\beta\gamma}{x} + \frac{2p^2 - j(j+1) - \alpha + \beta + \gamma + 2\alpha\beta + 2\alpha\gamma}{x-1}\right) g = 0. \quad (7)$$

With requirements α, β, γ

$$\alpha^2 + p^2 = 0 , \qquad \beta = 0,\ 2 , \qquad \gamma^2 + p^2 = 0 \quad (8)$$

equation (7) becomes simpler

$$\frac{d^2g}{dx^2} + \left(2\gamma - \frac{1-2\beta}{x} + \frac{1+2\alpha}{x-1}\right)\frac{dg}{dx} + \quad (9)$$
$$+ \left(\frac{j(j+1)+\alpha-\beta-\gamma-2\alpha\beta+2\beta\gamma}{x} + \frac{2p^2 - j(j+1) - \alpha + \beta + \gamma + 2\alpha\beta + 2\alpha\gamma}{x-1}\right) g = 0.$$

It can be identified as the confluent Heun equation (see [3, 4, 5]) for $H(A,B,C,D,F;x)$

$$\frac{d^2H}{dx^2} + \left(A + \frac{1+B}{x} + \frac{1+C}{x-1}\right)\frac{dH}{dx} + \tag{10}$$
$$+\left(\frac{1}{2}\,\frac{A-B-C+AB-BC-2F}{x} + \frac{1}{2}\,\frac{A+B+C+AC+BC+2D+2F}{x-1}\right)H = 0$$

with parameters

$$A = 2\gamma\,, \qquad B = 2\beta - 2\,, C = 2\alpha\,, \qquad D = 2p^2\,, \qquad F = 1 - j(j+1)\,.$$

Let it be $\beta = 0, \alpha = +\gamma = \pm ip$ (the sign $\pm$ will be set in the symbol p):

$$G^1_p(x) = (x-1)^{ip}\, e^{ipx}\; H(2ip, -2, 2ip, 2p^2, 1 - j(j+1);\; x)\,, \tag{11}$$

Let it be $\beta = 0, \alpha = -\gamma = \pm ip$:

$$G^2_p(x) = (x-1)^{ip}\, e^{-ipx}\; H(-2ip, -2, +2ip, 2p^2, 1 - j(j+1);\; x)\,. \tag{12}$$

Let $\beta = 2, \alpha = \gamma = ip$:

$$F^1_p(x) = (x-1)^{ip}\, e^{ipx}\; x^2\; H(2ip,\; 2,\; 2ip,\; 2p^2,\; 1 - j(j+1);\; x)\,, \tag{13}$$

Let $\beta = 2, \alpha = -\gamma = ip$:

$$F^2_p(x) = (x-1)^{ip}\, e^{ipx}\; x^2\; H(-2ip,\; 2,\; 2ip,\; 2p^2,\; 1 - j(j+1);\; x)\,, \tag{14}$$

2 Qualitative analysis of the problem

With the substitution

$$G(x) = \varphi\; F(x), \qquad \varphi = \sqrt{\frac{x}{x-1}}\;; \tag{15}$$

we produce an equation for $F(x)$ in the form

$$F'' + P^2(x)F = 0\,. \tag{16}$$

where an effective squared momentum $P^2(x)$ is given by (the notation is used: $j(j+1) = J$)

$$P^2 = \frac{4\,p^2x^4 - 4Jx^2 + 4(J+1)x - 3}{4\,x^2\,(x-1)^2}\,, \tag{17}$$

Let us examine possible turning points, roots of the equation

$$4p^2x^4 - 4Jx^2 + 4(J+1)x - 3 = 0\,. \tag{18}$$

From identity

$$4p^2x^4 - 4Jx^2 + 4(J+1)x - 3 = 4p^2(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

it follows

$$-\frac{3}{4p^2} = (x_1x_2)\,(x_3x_4)\;, \qquad -\frac{4(J+1)}{4p^2} = (x_1+x_2)x_3x_4 + x_1x_2(x_3+x_4)\;, \tag{19}$$
$$-\frac{4J}{4p^2} = x_1x_2 + (x_1+x_2)(x_3+x_4) + x_3x_4\;, \qquad 0 = (x_1+x_2)+(x_3+x_4)\;.$$

For the roots x_1, x_2, x_3, x_4 there are possible three variants:

$$-,\ +,\ +, + \;; \qquad -,\ -,\ -,\ + \;; \qquad -i,\ +i,\ -,\ + \;.$$

Asymptotic behavior of $P^2(x)$ is given by

$$x \to 1 \pm 0, \qquad P^2(x) \sim \frac{p^2 + 1/4}{(x-1)^2} \to +\infty\;; \tag{20}$$
$$x \to +\infty, \qquad P^2(x) \sim p^2\;; x \to +0, \qquad P^2(x) \sim -\frac{3}{4x^2} \to -\infty\;.$$

The most interesting from physical viewpoint is the situation when

$$x_1 < 0\;, \qquad 0 < x_2 < 1\;, \qquad x_4 > x_3 > 1\;. \tag{21}$$

The numerical analysis for P^2 depends of the values of parameter p (for definiteness, let $j = 1$). For illustrating, below we specify the following two cases:

1. If p is very close to 1 like: $p = \sqrt{\frac{37537.5}{37538}}$ the equation $P^2 = 0$ has 2 turning points: one positive and one negative (they both belong to non-physical region of the variable x), see figure [1]

   $$x_1 = 0.3115796719, x_2 = -1.936336591, x_{3,4} = 0.8123784595 \pm 0.7636574912i$$

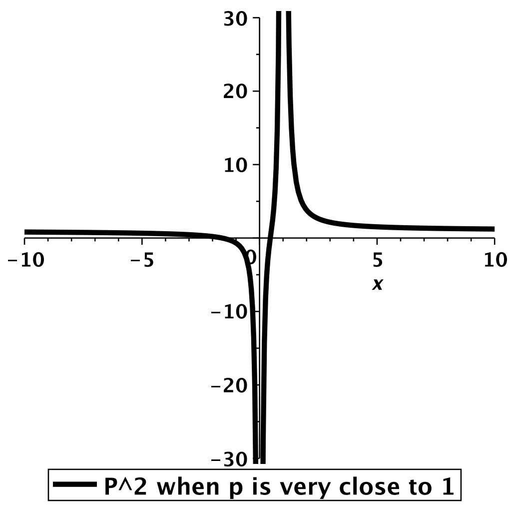

Figure 1: The variation of function P^2 in the vecinity of 1

The Figure 1 corresponds to situation when the all domain $x > +1$ is allowed for classical motion, $P^2(x) > 0$, and this is a situation when no tunneling effect arises.

2. If $p = \frac{1}{3}$, then all roots are real: 1 negatives and the other three positive, see figure [2]:

$$x_1 = 0.316345318, x_2 = 1.368216154, x_3 = 3.195604634, -4.880166106$$

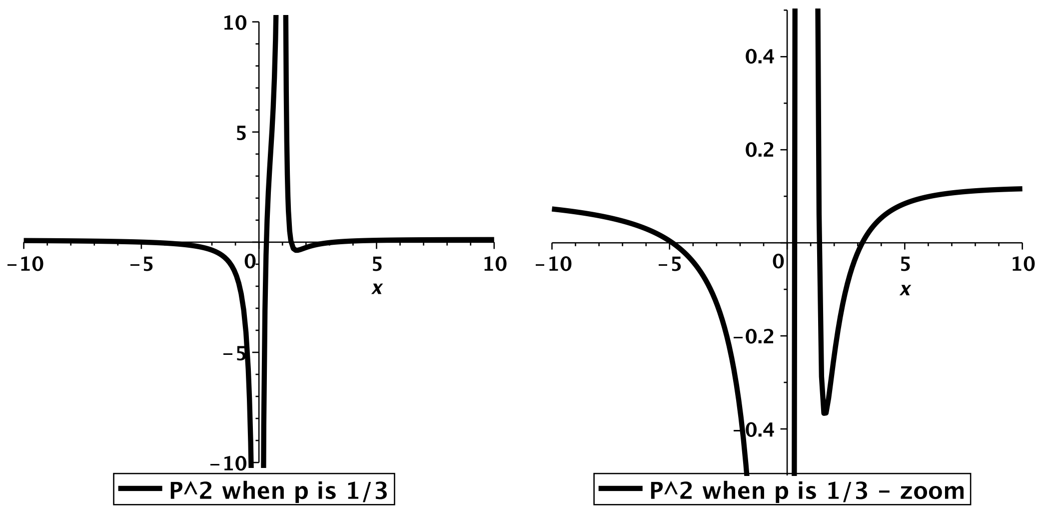

Figure 2: The variation of function P^2 when $p = \frac{1}{3}$

The Figure 2 corresponds to a situation when the physical domain $x > 1$ is divided into three parts: one forbidden and two ones allowed for classical motion.

One can obtain description which is simpler to interpret physically. It is suffices to introduce a new variable y

$$y = \ln(x-1), \qquad \frac{d}{dx} = \frac{1}{x-1}\frac{d}{dy}, \qquad \frac{d^2}{dx^2} = \frac{1}{(x-1)^2}\left(\frac{d^2}{dy^2} - \frac{d}{dy}\right) \; ;$$

equation (18) leads to

$$\left(\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{4p^2x^4 - 4Jx^2 + 4(J+1)x - 3}{4\,x^2}\right) F(y) = 0 \; , \qquad x = 1 + e^y \; . \qquad (22)$$

In the vicinity of the point $x = 1 + 0$ ($y \to -\infty$), equation (22) behaves as follows

$$\left(\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{4\,p^2 + 1}{4}\right) F(y) = 0 \; , \qquad (23)$$

with solutions in the form

$$F(y)_{y \to -\infty} \; = \; e^{y/2}\, e^{\pm i p y} \; . \qquad (24)$$

In fact, the term $e^{y/2}$ is compensated by previously introduced term

$$\varphi(x) = \sqrt{\frac{x}{x-1}}\bigg|_{x\to 1+0} = \frac{1}{e^{y/2}} = e^{-y/2}\ . \tag{25}$$

In other words, it means that equation (16) can be written with the use of y as follows

$$\left(\frac{d^2}{dy^2} + \frac{4p^2x^4 - 4Jx^2 + 4(J+1)x - 3}{4\,x^2}\right) G(y) = 0\ , \qquad x = 1 + e^y\ . \tag{26}$$

In the variable y, physical region for motion $x \in (+1, +\infty)$ is stretched into the interval $y \in (-\infty, +\infty)$ and schematically the problem clarified by In the case when the function P^2 depends on variable y, such as:

$$P^2(y) = \frac{4p^2(1+e^y)^4 - 4J(1+e^y)^2 + (4(J+1))(1+e^y) - 3}{4(1+e^y)^2(e^y)^2}$$

we obtain the following behavior of this function, for a subunitary value of the parameter $p = \frac{1}{6000}$. In this case there are two turning points:

$$y_3 = -0.6931491806; y_4 = 8.699181328\ :$$

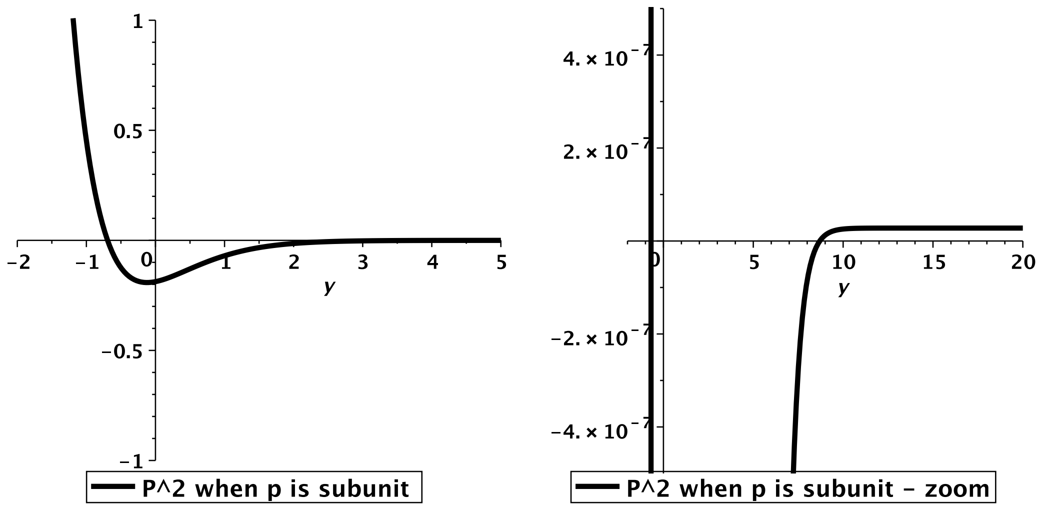

Figure 3: The variation of the function P^2 when $p < 1$

The Figure 3 corresponds to a situation when the physical domain $y \in (-\infty, +\infty)$ is divided into three parts: one forbidden and two ones allowed for classical motion. Let us solve analytically the task of finding turning points, the roots of the equation

$$P^2 = \frac{4\,p^2x^4 - 4Jx^2 + 4(J+1)x - 3}{4\,x^2\,(x-1)^2} = 0\,, \tag{27}$$

so we get 4-nd order equation

$$4\,p^2x^4 - 4Jx^2 + 4(J+1)x - 3 = 0\ , \qquad J = j(j+1)\ , \qquad j = 0,1,2,3,...$$

or

$$x^4 + Ax^2 + Bx + C = 0\ , \tag{28}$$

where (introduce notation $1/p^2 = \sigma$)

$$A = -J\sigma\ , \qquad B = (J+1)\sigma, \qquad C = -\frac{3}{4}\sigma\ . \tag{29}$$

According the Cartesian-Euler method, one should firstly solve a 3-d order equation

$$z^3 + \frac{A}{2}z^2 + \frac{A^2 - 4C}{16}z - \frac{B^2}{64} = 0 \tag{30}$$

and four roots x_1, x_2, x_3, x_4 of equation (28) are to be chosen from

$$\begin{array}{c} +\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}\ ,-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}\ ,+\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}\ ,+\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}\ , \\ -\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}\ ,-\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}\ ,+\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}\ ,-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}\ , \end{array} \tag{31}$$

with requirement

$$\sqrt{z_1}\ \sqrt{z_2}\ \sqrt{z_3} = -\frac{B}{8} = -\frac{(J+1)\sigma}{8} = -\frac{(J+1)}{8p^2} < 0\ . \tag{32}$$

So, when z_1, z_2, z_3 are real, the real four roots are

$$-\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}\ ,+\sqrt{z_1}-\sqrt{z_2}+\sqrt{z_3}\ ,+\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3}\ ,-\sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3}\ . \tag{33}$$

In turn, when

$$z_1 = a_1\ , \qquad z_2 = a + ib\ , \qquad z_3 = a - ib\ ,$$

the four roots are

$$\begin{array}{c} -\sqrt{a_1}+\sqrt{a+ib}+\sqrt{a+ib}^{*}-\ ,-\sqrt{a_1}-\sqrt{a+ib}-\sqrt{a+ib}^{*} - \text{real root}\ , \\ +\sqrt{a_1}+\sqrt{a+ib}-\sqrt{a+ib}^{*}-,+\sqrt{a_1}-\sqrt{a+ib}+\sqrt{a+ib}^{*} - \text{complex root}\ . \end{array} \tag{34}$$

Allowing for

$$\frac{A}{2} = -\frac{J\sigma}{2}\ , \qquad \frac{A^2-4C}{16} = \frac{J^2\sigma^2+3\sigma}{16}\ , \qquad -\frac{B^2}{64} = -\frac{(J+1)^2\sigma^2}{64}\ ,$$

we get

$$x^3 - \frac{J\sigma}{2}x^2 + \frac{J^2\sigma^2+3\sigma}{16}x - \frac{(J+1)^2\sigma^2}{64} = 0 \Leftrightarrow x^3 + rx^2 + sx + t = 0\ . \tag{35}$$

Changing the variable

$$y = x + \frac{r}{3} = x - \frac{J\sigma}{6}\ , \tag{36}$$

we obtain a cubic equation in reduced form

$$y^3 + \Pi y + Q = 0 \, , \qquad \Pi = \frac{3s - r^2}{3} = -\frac{1}{48} J^2 \sigma^2 + \frac{3\sigma}{16} \, , \qquad (37)$$
$$Q = \frac{2r^3}{27} - \frac{rs}{3} + t = \sigma^2 \left(\frac{\sigma J^3}{864} - \frac{J^2}{64} - \frac{1}{64} \right) .$$

It should be noted that for a special case $j = 0, J = 0$, the cubic equation reads

$$y^3 + \Pi y + Q = 0 \, , \qquad \Pi = \frac{3\sigma}{16} > 0 \, , \qquad Q = -\frac{\sigma^2}{64} \, . \qquad (38)$$

For $J \neq 0$, the quantity Π can be either negative

$$\Pi < 0 \qquad \Longrightarrow \qquad 1 < \sigma \, \frac{J^2}{9}, \qquad p^2 < \frac{J^2}{9} \, , \qquad (39)$$

or positive

$$\Pi > 0 \qquad \Longrightarrow \qquad 1 > \sigma \, \frac{J^2}{9}, \qquad p^2 > \frac{J^2}{9} \, . \qquad (40)$$

The structure of roots is determined by the sign of the discriminant $D = \left(\frac{\Pi}{3}\right)^3 + \left(\frac{Q}{2}\right)^2$. When $D < 0$ (note that when $J = 0$, the discriminant D is positive), all roots of the cubic equation (38) are real: $\left(\frac{Q}{2}\right)^2 < -\left(\frac{\Pi}{3}\right)^3$; this is possible only at $\Pi < 0$. Let us find an explicit expression for D:

$$D = \frac{1}{16384} \sigma^4 (J^4 + 1) - \frac{1}{110592} \sigma^5 J^3 (J^2 - J + 1) + \frac{1}{24576} \sigma^3 (\sigma \, J^2 + 6) \, .$$

After transition from σ to $1/k^2$:

$$D = \frac{1}{p^{10}} \frac{1}{4096} \left[p^4 + p^2 \frac{(J^4 + 1)}{4} + k^2 \frac{J^2}{6} - \frac{1}{27} J^3 (J^2 - J + 1) \right] . \qquad (41)$$

Inequality $D < 0$ leads to

$$p^4 + 2 \, p^2 \frac{3J^4 + 2J^2 + 3}{24} - \frac{J^5 - J^4 + J^3}{27} < 0 \, , \qquad (42)$$

or

$$\left(p^2 + \frac{3J^4 + 2J^2 + 3}{24} \right)^2 < \left(\frac{3J^4 + 2J^2 + 3}{24} \right)^2 + \frac{J^5 - J^4 + J^3}{27} \; ;$$

so that $D(p^2, J) < 0$ is equivalent to

$$0 < p^2 < -\frac{3J^4 + 2J^2 + 3}{24} + \sqrt{ \left(\frac{3J^4 + 2J^2 + 3}{24} \right)^2 + \frac{J^5 - J^4 + J^3}{27} } \, . \qquad (43)$$

With notation

$$\pi^2(J) = -\frac{3J^4+2J^2+3}{24} + \sqrt{\left(\frac{3J^4+2J^2+3}{24}\right)^2 + \frac{J^5-J^4+J^3}{27}}\ , \tag{44}$$

condition of having three real roots ir written as

$$p^2 < \pi^2(J)\ . \tag{45}$$

Schematically, we have

$$\pi^2(J) = -a + \sqrt{a^2+b} = a(-1+\sqrt{1+\frac{b}{a^2}})\ , \qquad \frac{b}{a^2} = \frac{24\cdot 24(J^5-J^4+J^3)}{27(3J^4+2J^2+3)^2}\ .$$

For large J we derive approximate relation

$$\pi^2(J) = -a+\sqrt{a^2+b} = a(-1+\sqrt{1+\frac{b}{a^2}}) \approx +\frac{b}{2a} = \frac{24(J^5-J^4+J^3)}{2\cdot 27(3J^4+2J^2+3)} \approx \frac{4}{9}J\ . \tag{46}$$

The graphic for $\pi^2(J)$ is given in Fig. 4 (also see the table below).

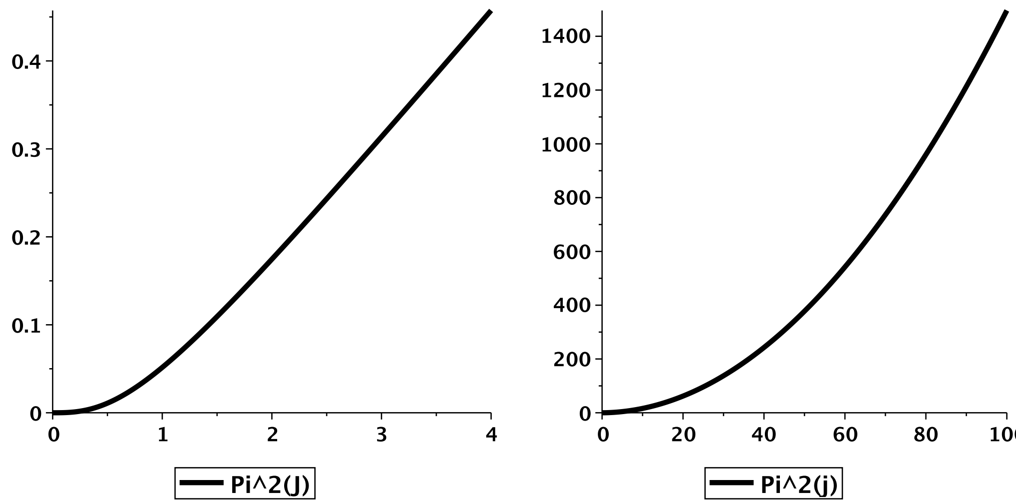

Figure 4: Left side: Behavior of $\pi^2(J)$; Right side: The graphic for $\pi^2(j)$

The physical values for J are

$$J = 0,\ \ 1\cdot 2,\ \ 2\cdot 3,\ \ 3\cdot 4,\ \ 4\cdot 5,\ \ 5\cdot 6, .., \ \ 50\cdot 51, ..., \ \ 100\cdot 101, ..., \ \ 300\cdot 300, ...$$

Table 1: The values numerically generated

J	$\pi(J)$	$\pi^2(J)$	$\frac{4}{9}J$	$\frac{J}{3}$
0	0	0	0	0
$1\cdot 2$	0.4178411339	0.174591213	0.8888888889	0.6666666667
$2\cdot 3$	0.8655902250	0.7492464	2.666666667	2
$3\cdot 4$	1.278208708	1.633817	5.333333333	4
$4\cdot 5$	1.678485180	2.81732	8.888888889	6.666666667
$5\cdot 6$	2.073146763	4.2979	13.33333333	10
$6\cdot 7$	2.464802223	6.0753	18.66666667	14
$7\cdot 8$	2.854698856	8.149	24.88888889	18.66666667
$8\cdot 9$	3.243262248	10.519	32	24
...	...	...	...	...
$10\cdot 11$	4.018187817	16.15	48.88888889	36.66666667
...	...	...	...	...
$50\cdot 51$	19.43	377.6289	1133.333333	850
...	...	...	...	...
$100\cdot 101$	38.68	1496	4488.888889	3366.666667
$101\cdot 102$	39.06	1526	4578.666667	3434
$102\cdot 103$	39.44	1556.3	4669.333333	3502
$200\cdot 201$ ...	77.14 ...	5952 ...	17866.66 ...	13400 ...
$300\cdot 301$	119.73	14336	40133	30100

Condition for existing four real roots in equation (18) at the parameter restrictions for $p,\ j$ we will find with the help of the command [6]

$$\text{Reduce}[\frac{4p^2x^4 - 4j(j+1)x^2 + 4(j(j+1)+1)x - 3}{4x^2(x-1)^2} == 0\&\&p > 0\&\&j \geq 1, x, \text{Reals}]$$

From whence it follows

$$0 < p < \text{Root}[-4j^3 - 8j^4 - 24j^7 - 36j^8 - 20j^9 - 4j^{10} +$$
$$(27 + 18j^2 + 36j^3 + 45j^4 + 108j^5 + 162j^6 + 108j^7 + 27j^8)\#1^2 + 108\#1^4\&, 2]\&\&$$
$$x == \text{Root}[(4 + 4j + 4j^2)\#1 - (4j + 4j^2)\#1^2 - 3 + 4p^2\#1^4\&, i]\ (i = 1, 2, 3, 4).$$

Visualization is reached with the help of the command

$$\text{Manipulate}[\text{Plot}[\text{Table}[\text{Root}[(4 + 4j + 4j^2)\#1 - (4j + 4j^2)\#1^2 - 3 + 4p^2\#1^4\&, i],$$
$$\{i, 1, 4\}], \{p, 0, \text{Root}[-4j^3 - 8j^4 - 24j^7 - 36j^8 - 20j^9 - 4j^{10} +$$
$$(27 + 18j^2 + 36j^3 + 45j^4 + 108j^5 + 162j^6 + 108j^7 + 27j^8)\#1^2 + 108\#1^4\&, 2]\},$$
$$\text{PlotRange} \to \{\{0, 0.45\}, \{-10, 10\}\}, \text{AxesStyle} \to \text{Directive}[\text{Black}, 14],$$

$$\text{PlotStyle} \to \text{Directive}[\text{Black}, \text{Thickness} \to 0.015],\ \text{AxesLabel} \to \{\text{Style}[p, 16, \text{Bold}],$$

Style["$x(p)$", 16, Bold]}], {{j, 1, TraditionalForm[j]}, 1, 10, 1}]

In the Fig.5 the case $j = 1$ is specified

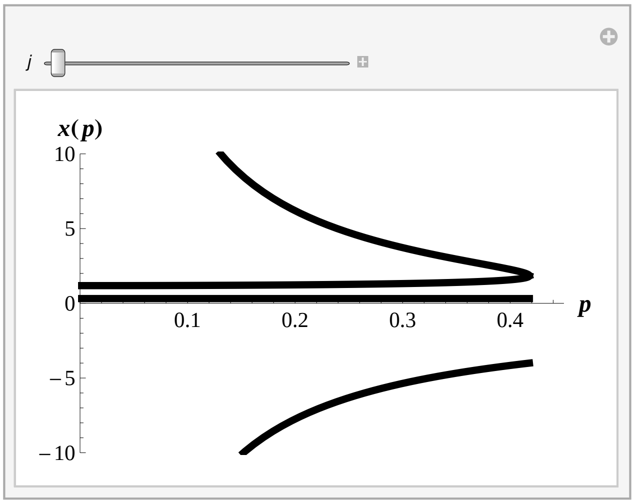

Figure 5: Curves of the four roots as functions of the parameter p

DISCISSION: The problem of description of various fields in the Schwarzschild black hole background always is reduced to an ordinary differential equation of the Heun type with four singular points, which is minimal generalization of a master equation providing us to mathematically describe quantum mechanical problems in potentials of barrier type. Exact condition insuring existence of two turning points and thereby possibility to have quantum mechanical tunneling effect are found analytically and examined numerically; which is substantial at further numerical examination of possible solution of the differential equation under consideration and thereby the black hole dynamics.

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