On generating systems of some transformation semigroups of the boolean

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Let M be a poset with a partial order $\leq$. A transformation $\alpha : M \to M$ is called *order-decreasing* if $\alpha(x) \leq x$ for all $x \in \mathrm{dom}(\alpha)$. The set of such transformations is denoted by $\mathcal{F}(\mathcal{M})$. A transformation α is called *order-preserving* if for every $x, y \in \mathrm{dom}(\alpha)$, $x \leq y$ implies $\alpha(x) \leq \alpha(y)$. The set of such transformations is denoted by $\mathcal{O}(\mathcal{M})$. We consider a subset $\mathcal{C}(\mathcal{M}) = \mathcal{F}(\mathcal{M}) \cap \mathcal{O}(\mathcal{M})$. The sets $\mathcal{F}(\mathcal{M})$, $\mathcal{O}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ are semigroups with respect to the composition of transformations.

Many authors (see [1] and references therein) studied the semigroups $\mathcal{F}(\mathcal{M})$, $\mathcal{O}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ in the case where the order $\leq$ on M is linear. We deal with the set of all subsets of a n–element set $N = \{1, 2, \ldots, n\}$ naturally ordered by inclusion, that is, with the boolean $\mathcal{B}_n$.

We focus on three classic semigroups: the symmetric semigroup of all transformations of the set $\mathcal{B}_n$; the semigroup of all partial transformations of the set $\mathcal{B}_n$ and the symmetric inverse semigroup of all partial injective transformations of $\mathcal{B}_n$. Thus, we have nine semigroups of order-consistent transformations of the set $\mathcal{B}_n$: $\mathcal{F}(\mathcal{B}_n)$, $\mathcal{PF}(\mathcal{B}_n)$, $\mathcal{IF}(\mathcal{B}_n)$, $\mathcal{O}(\mathcal{B}_n)$, $\mathcal{PO}(\mathcal{B}_n)$, $\mathcal{IO}(\mathcal{B}_n)$, $\mathcal{C}(\mathcal{B}_n)$, $\mathcal{PC}(\mathcal{B}_n)$, $\mathcal{IC}(\mathcal{B}_n)$.

Denote by $J(S)$ the set of idempotents of defect 1 of a transformation semigroup S. Let ϵ be an identity transformation of semigroup S.

Our main results are the following:

Theorem 1. *The symmetric semigroup of order-decreasing transformations $\mathcal{F}(\mathcal{B}_n)$ is generated by the set $J(\mathcal{F}(\mathcal{B}_n)) \bigcup \{\epsilon\}$, where $|J(\mathcal{F}(\mathcal{B}_n))| = 3^n - 2^n$. The semigroup of all partial order-decreasing transformations $\mathcal{PF}(\mathcal{B}_n)$ is generated by the set $J(\mathcal{PF}(\mathcal{B}_n)) \bigcup \{\epsilon\}$, where $|J(\mathcal{PF}(\mathcal{B}_n))| = 3^n$.*

Theorem 2. *The semigroups $\mathcal{IF}(\mathcal{B}_n)$, $\mathcal{O}(\mathcal{B}_n)$ (for $n \geq 2$), $\mathcal{PO}(\mathcal{B}_n)$ (for $n \geq 2$), $\mathcal{IO}(\mathcal{B}_n)$, $\mathcal{C}(\mathcal{B}_n)$ (for $n \geq 3$), $\mathcal{PC}(\mathcal{B}_n)$ (for $n \geq 3$), $\mathcal{IC}(\mathcal{B}_n)$ are not generated by the idempotents.*

References

[1] *Ganyushkin O., Mazorchuk V.* Classical Finite Transformation semigroups. An Introduction. — Algebra and Applications. — London: Springer–Verlag, 2009. — **9**. — XII, 314 p.

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