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APPROXIMATION OF (ψ, β)-DIFFERENTIABLE FUNCTIONS BY POISSON INTEGRALS IN THE UNIFORM METRIC

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We obtain asymptotic equalities for upper bounds of approximations of functions from the class $C^{\psi}_{\beta,\infty}$ by Poisson integrals in the metric of the space C.

1. Statement of the Problem and Some Historical Information

Let C be the space of 2π-periodic continuous functions with the norm

$$\|f\|_C = \max_t |f(t)|,$$

let L_∞ be the space of 2π-periodic, measurable, essentially bounded functions with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|,$$

and let L_1 be the space of 2π-periodic summable functions with the norm

$$\|f\|_{L_1} = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)| dt.$$

In 1983, Stepanets proposed a new approach to the classification of periodic functions. This approach is based on the notion of (ψ, β)-derivative (see, e.g., [1–4]). The classes L^{ψ}_{β} of functions $f \in L_1$ are introduced as follows: Let a sequence $\psi = \psi(k)$ and parameter β be such that the series

$$\sum_{k=1}^{\infty} \psi(k) \cos\left(kt + \frac{\beta\pi}{2}\right) \tag{1}$$

is the Fourier series of a certain summable function $\Psi_\beta(t)$. Then the following equality holds for any $f \in L^{\psi}_{\beta}$ and almost all $x \in R$:

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x+t)\, \Psi_\beta(t) dt,$$

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where $\varphi(\cdot)$ is a certain function from L_1 and

$$\int_{-\pi}^{\pi} \varphi(t)dt = 0.$$

The function φ is called the (ψ, β)-derivative of the function f and is denoted by f_β^ψ.

If $f \in L_\beta^\psi$ and, in addition, $f_\beta^\psi \in \mathfrak{N}$, $\mathfrak{N} \subseteq L_1$, then one says that $f \in L_\beta^\psi \mathfrak{N}$. The subsets of continuous functions from L_β^ψ and $L_\beta^\psi \mathfrak{N}$ are denoted by C_β^ψ and $C_\beta^\psi \mathfrak{N}$, respectively. Further, if $\mathfrak{N}$ coincides with the unit ball of the space L_∞, i.e.,

$$\mathfrak{N} = \left\{ f_\beta^\psi \in L_\infty \colon \operatorname*{ess\,sup}_t \left| f_\beta^\psi(t) \right| \le 1 \right\},$$

then the classes $C_\beta^\psi \mathfrak{N}$ are denoted by $C_{\beta,\infty}^\psi$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C_{\beta,\infty}^\psi$ coincide with the classes $W_{\beta,\infty}^r$, and $f_\beta^\psi(x) = f_\beta^{(r)}(x)$ is the Weyl–Nagy (r, β)-derivative [5]. Furthermore, if $\beta = r$, $r \in N$, then f_β^ψ is the rth-order derivative of the function f, and the classes $C_{\beta,\infty}^\psi$ are the well-known Sobolev classes W_∞^r.

Following Stepanets (see, e.g., [4, p. 155]), we denote by $\mathfrak{M}$ the set of all convex-downward sequences $\psi(k)$ for which

$$\lim_{k\to\infty} \psi(k) = 0.$$

If a sequence $\psi(k)$ satisfies the conditions $\psi \in \mathfrak{M}$ and

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} < \infty,$$

then, by virtue of Theorem 1.7.3 in [3, p. 28], series (1) is the Fourier series of the function $\Psi_\beta(t)$.

Without loss of generality, we can assume that the sequences $\psi(k)$ from the set $\mathfrak{M}$ are restrictions of certain positive, continuous, convex-downward functions $\psi(t)$ of a continuous argument $t \ge 1$ that vanish at infinity to the set of natural numbers. The set of these functions is also denoted by $\mathfrak{M}$. Thus, in what follows,

$$\mathfrak{M} = \left\{ \psi(t) \colon \; \psi(t) > 0, \; \psi(t_1) - 2\psi\left((t_1 + t_2)/2\right) + \psi(t_2) \ge 0 \;\; \forall t_1, \; t_2 \in [1, \infty), \; \lim_{t\to\infty} \psi(t) = 0 \right\}.$$

Let $\mathfrak{M}'$ denote the set of functions $\psi \in \mathfrak{M}$ for which

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty.$$

In the set $\mathfrak{M}$, we select a subset $\mathfrak{M}_0$ as follows (see, e.g., [4, p. 160]):

$$\mathfrak{M}_0 = \left\{ \psi \in \mathfrak{M} \colon \; 0 < \frac{t}{\eta(t) - t} \le K \;\; \forall t \ge 1 \right\},$$

where

$$\eta(t) = \eta(\psi, t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right),$$

ψ^{-1} is the function inverse to ψ, and K is a constant that may depend on ψ.

Let $f \in L_1$. The quantity

$$P_\delta(f; x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-k/\delta}\left(a_k \cos kx + b_k \sin kx\right), \quad \delta > 0,$$

where a_0, a_k, and b_k are the Fourier coefficients of the function f, is called the Poisson integral (see, e.g., [6, p. 161]).

In the present paper, we study the asymptotic behavior of the quantity

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \left\| f(\cdot) - P_\delta(f; \cdot)\right\|_C \tag{2}$$

as $\delta \to \infty$.

If we determine the explicit form of a function $\varphi(\delta) = \varphi(\mathfrak{N}; \delta)$ such that

$$\mathcal{E}\left(\mathfrak{N}; P_\delta\right)_X = \varphi(\delta) + o\left(\varphi(\delta)\right) \quad \text{as} \quad \delta \to \infty,$$

then, following Stepanets [4, p. 198], we say that the Kolmogorov–Nikol'skii problem for the Poisson integral $P_\delta(f; x)$ is solved on the class $\mathfrak{N}$ in the metric of the space X.

Note that the Kolmogorov–Nikol'skii problem for the functions $P_\delta(f; x)$ on the Sobolev classes W^1_∞ was solved by Natanson in [7]. In [8], Timan determined the exact values of the upper bounds of deviations of Poisson integrals from functions of the class W^r_∞, $r > 0$. A solution of the Kolmogorov–Nikol'skii problem on the class $W^r_{\beta,\infty}$, $r > 0$, $\beta \in R$, was obtained by Bausov in [9]. In particular, he obtained the following asymptotic equality for the class $W^1_{\beta,\infty}$:

$$\mathcal{E}\left(W^1_{\beta,\infty}; P_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\frac{\ln\delta}{\delta} + O\left(\frac{1}{\delta}\right), \quad \delta \to \infty. \tag{3}$$

Approximation properties of the method of approximation by Poisson integrals on other classes of differentiable functions were studied in [10, 11].

2. Some Estimates for Fourier Integrals

To investigate the asymptotic behavior of (2) as $\delta \to \infty$, it is necessary to establish conditions under which the Fourier transform

$$\hat{\tau}(t) = \hat{\tau}_\delta(t) = \frac{1}{\pi}\int_0^{\infty} \tau(u)\cos\left(ut + \frac{\beta\pi}{2}\right) du \tag{4}$$

of the function $\tau(\cdot)$ defined by the relation

$$\tau(u) = \tau_\delta(u;\psi) = \begin{cases} (1-e^{-u})\,\dfrac{\psi(1)}{\psi(\delta)}, & 0 \le u \le \dfrac{1}{\delta}, \\ (1-e^{-u})\,\dfrac{\psi(\delta u)}{\psi(\delta)}, & u \ge \dfrac{1}{\delta}, \end{cases} \tag{5}$$

is summable on the entire number axis.

To analyze this problem, we need the statements presented below.

Definition 1 [9, p. 18]. *Suppose that a function* $\tau(u)$ *is defined on* $[0,\infty)$ *and absolutely continuous and* $\tau(\infty) = 0$. *One says that the function* $\tau(u)$ *belongs to* $\mathcal{E}_1$ *if the definition of the derivative* $\tau'(u)$ *can be extended to the points where it does not exist so that the following integrals exist:*

$$\int_0^{1/2} u|d\tau'(u)| \qquad \textit{and} \qquad \int_{1/2}^{\infty} |u-1||d\tau'(u)|.$$

Proposition 1 [9, p. 19]. *If* $\tau(u) \in \mathcal{E}_1$, *then*

$$|\tau(u)| \le H(\tau), \tag{6}$$

where

$$H(\tau) = |\tau(0)| + |\tau(1)| + \int_0^{1/2} u\left|d\tau'(u)\right| + \int_{1/2}^{\infty} |u-1|\left|d\tau'(u)\right|. \tag{7}$$

Proposition 2 [4, p. 161]. *A function* $\psi \in \mathfrak{M}$ *belongs to* $\mathfrak{M}_0$ *if and only if the value*

$$\alpha(t) = \frac{\psi(t)}{t\,|\psi'(t)|}, \qquad \psi'(t) = \psi'(t+0), \tag{8}$$

satisfies the condition $\alpha(t) \ge K > 0 \ \ \forall t \ge 1$.

Proposition 3 [4, p. 175]. *A function* $\psi \in \mathfrak{M}$ *belongs to* $\mathfrak{M}_0$ *if and only if, for an arbitrary fixed number* $c > 1$, *there exists a constant* K *such that the following inequality holds for all* $t \ge 1$:

$$\frac{\psi(t)}{\psi(ct)} \le K.$$

In what follows, K and K_i denote constants that are, generally speaking, different.

We set $\mathfrak{M}_0' = \mathfrak{M}_0 \cap \mathfrak{M}'$. The following statement is true:

Lemma 1. *Suppose that* $\psi\in\mathfrak{M}_0'$ *and the function* $g(u)=u\psi(u)$ *is convex upward or downward on* $[b,\infty)$, $b\geq 1$. *Then, for the function* $\tau(\cdot)$ *defined by (5), its Fourier transform of the form (4) is summable on the entire number axis, i.e., the integral*

$$A(\tau)=\int_{-\infty}^{\infty}\left|\hat{\tau}_\delta(t)\right|dt,\quad \delta\to\infty, \tag{9}$$

is convergent.

Proof. To verify the convergence of integral (9), according to Theorem 1 in [9] we estimate the integrals

$$\int_0^{1/2}u\left|d\tau'(u)\right|,\qquad \int_{1/2}^{\infty}|u-1|\left|d\tau'(u)\right|, \tag{10}$$

$$\left|\sin\frac{\beta\pi}{2}\right|\int_0^{\infty}\frac{|\tau(u)|}{u}du,\qquad \int_0^1\frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{11}$$

To estimate the first integral in (10), we divide the segment $[0;1/2]$ into two parts: $[0;1/\delta]$ and $[1/\delta;1/2]$ (for $\delta>2b$). Since $\tau''(u)<0$ on $[0,1/\delta]$, taking into account that

$$1-e^{-u}<u,\quad u\geq 0, \tag{12}$$

we obtain

$$\int_0^{1/\delta}u\left|d\tau'(u)\right|=\frac{\psi(1)}{\psi(\delta)}\left(1-\frac{1}{\delta}e^{-1/\delta}-e^{-1/\delta}\right)=O\left(\frac{1}{\delta^2\psi(\delta)}\right),\quad \delta\to\infty. \tag{13}$$

Now let $u\in[1/\delta;1/2]$. We set $\tau(u)=\tau_1(u)+\tau_2(u)$, where

$$\tau_1(u)=\left(1-e^{-u}-u\right)\frac{\psi(\delta u)}{\psi(\delta)}, \tag{14}$$

$$\tau_2(u)=u\frac{\psi(\delta u)}{\psi(\delta)}. \tag{15}$$

Then

$$\int_{1/\delta}^{1/2}u|d\tau'(u)|\leq\int_{1/\delta}^{1/2}u|d\tau_1'(u)|+\int_{1/\delta}^{1/2}u|d\tau_2'(u)|,\quad \delta>2. \tag{16}$$

Let us estimate the first integral on the right-hand side of (16). To this end, first, we investigate the function

$$\overline{\mu}(u)=1-e^{-u}-u. \tag{17}$$

It follows from the relations $\overline{\mu}'(u) = e^{-u} - 1$, $\overline{\mu}''(u) = -e^{-u}$, $\overline{\mu}(0) = 0$, and $\overline{\mu}'(0) = 0$ that, for $u \geq 0$, we have

$$\overline{\mu}(u) \leq 0, \qquad \overline{\mu}'(u) \leq 0, \qquad \overline{\mu}''(u) < 0. \tag{18}$$

Taking into account relations (18) and (12) and the fact that

$$e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad u \geq 0,$$

we obtain

$$\left|\overline{\mu}(u)\right| = u - 1 + e^{-u} \leq \frac{u^2}{2}, \qquad \left|\overline{\mu}'(u)\right| = 1 - e^{-u} \leq u, \qquad \left|\overline{\mu}''(u)\right| = e^{-u} \leq 1. \tag{19}$$

Since, for $u \geq 1/\delta$, according to (14) and (17), one has

$$\left|d\tau_1'(u)\right| \leq \left(\left|\overline{\mu}(u)\right|\frac{\delta^2\psi''(\delta u)}{\psi(\delta)} + 2\left|\overline{\mu}'(u)\right|\frac{\delta|\psi'(\delta u)|}{\psi(\delta)} + \left|\overline{\mu}''(u)\right|\frac{\psi(\delta u)}{\psi(\delta)}\right) du, \tag{20}$$

taking (19) into account we get

$$\int\limits_{1/\delta}^{1/2} u|d\tau_1'(u)| \leq \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} \frac{u^3}{2}\delta^2\psi''(\delta u)du + \frac{2}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du + \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u\psi(\delta u)du.$$

Integrating the first integral on the right-hand side of the last inequality by parts, we obtain

$$\int\limits_{1/\delta}^{1/2} u|d\tau_1'(u)| \leq \frac{1}{\psi(\delta)}\, \frac{u^3}{2}\delta\psi'(\delta u)\Bigg|_{1/\delta}^{1/2} + \frac{7}{2\psi(\delta)} \int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du + \frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u\psi(\delta u)du. \tag{21}$$

Using the conditions of Proposition 2, for $\psi \in \mathfrak{M}_0$ we obtain

$$\frac{1}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u^2\delta\left|\psi'(\delta u)\right| du \leq \frac{K}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u\psi(\delta u)du.$$

By virtue of Proposition 3, relation (21) yields

$$\int\limits_{1/\delta}^{1/2} u\left|d\tau_1'(u)\right| \leq K_1 + \frac{K_2}{\delta^2\psi(\delta)} + \frac{K_3}{\psi(\delta)} \int\limits_{1/\delta}^{1/2} u\psi(\delta u)du. \tag{22}$$

We consider the integral on the right-hand side of inequality (22) on the segments $[1/\delta, b/\delta]$ and $[b/\delta, 1/2]$, $\delta > 2b$. Since the function $g(u) = u\psi(u)$ is bounded on $[1, b]$, we have

$$\frac{1}{\psi(\delta)}\int_{1/\delta}^{b/\delta} u\psi(\delta u)du = \frac{1}{\delta^2\psi(\delta)}\int_{1}^{b} g(u)du \le \frac{K}{\delta^2\psi(\delta)}. \tag{23}$$

Further, since the function $g(u)$ is convex upward or downward for $u \ge b$ and $g(u) \ne 0$, the following two cases are possible for $u \in [b, \delta]$: either $u\psi(u) \le b\psi(b)$ or $u\psi(u) \le \delta\psi(\delta)$. Thus,

$$\frac{1}{\psi(\delta)}\int_{b/\delta}^{1/2} u\psi(\delta u)du = \frac{1}{\delta^2\psi(\delta)}\int_{b}^{\delta/2} g(u)du \le \frac{1}{\delta^2\psi(\delta)}\int_{b}^{\delta} g(u)du = O\left(1 + \frac{1}{\delta\psi(\delta)}\right) \quad \text{as} \quad \delta \to \infty. \tag{24}$$

With regard for (23) and (24), we obtain the following relation from (22):

$$\int_{1/\delta}^{1/2} u|d\tau_1'(u)| = O\left(1 + \frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty. \tag{25}$$

Let us estimate the second integral on the right-hand side of (16) on the segment $[1/\delta, b/\delta]$, $\delta > 2b$. It follows from (15) that

$$\tau_2''(u) = 2\delta\frac{\psi'(\delta u)}{\psi(\delta)} + \delta^2\frac{u\psi''(\delta u)}{\psi(\delta)}. \tag{26}$$

Using relation (26) and taking into account that the function $\psi(u)$ is decreasing and convex downward for $u \ge 1$, we obtain

$$\int_{1/\delta}^{b/\delta} u\left|d\tau_2'(u)\right| \le \frac{\delta^2}{\psi(\delta)}\int_{1/\delta}^{b/\delta} u^2\psi''(\delta u)du + \frac{2\delta}{\psi(\delta)}\int_{1/\delta}^{b/\delta} u\left|\psi'(\delta u)\right|du.$$

Since $\psi(\delta u) \le \psi(1)$ for $u \in [1/\delta, b/\delta]$, $\delta > 2b$, by virtue of Proposition 2 we obtain the following relation for a function $\psi \in \mathfrak{M}_0$:

$$\frac{\delta}{\psi(\delta)}\int_{1/\delta}^{b/\delta} u\left|\psi'(\delta u)\right|du \le \frac{K}{\psi(\delta)}\int_{1/\delta}^{b/\delta} \psi(\delta u)du \le \frac{K\psi(1)(b-1)}{\delta\psi(\delta)}.$$

Integrating by parts, we get

$$\frac{\delta^2}{\psi(\delta)}\int_{1/\delta}^{b/\delta} u^2\psi''(\delta u)du \le \frac{K_1}{\delta\psi(\delta)}.$$

Therefore,

$$\int\limits_{1/\delta}^{b/\delta} u|d\tau_2'(u)| \le \frac{K_2}{\delta\psi(\delta)}. \tag{27}$$

Let us estimate the second integral on the right-hand side of (16) on the segment $[b/\delta, 1/2]$, $\delta > 2b$. Since the function $g(u) = u\psi(u)$ is convex on $[b;\infty)$, we have

$$\int\limits_{b/\delta}^{1/2} u\left|d\tau_2'(u)\right| = \left|\int\limits_{b/\delta}^{1/2} u\,d\tau_2'(u)\right| = \left|\left(u\tau_2'(u) - \tau_2(u)\right)\Big|_{b/\delta}^{1/2}\right| = O\left(1 + \frac{1}{\delta\psi(\delta)}\right). \tag{28}$$

Using relations (13), (16), (25), (27), and (28), we obtain

$$\int\limits_{0}^{1/2} u|d\tau'(u)| = O\left(1 + \frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty. \tag{29}$$

We estimate the second integral in (10). For $u \in [1/\delta;\infty)$, according to (5), we have

$$\psi(\delta)d\tau'(u) = \left\{(1-e^{-u})\delta^2\psi''(\delta u) + 2\delta e^{-u}\psi'(\delta u) - e^{-u}\psi(\delta u)\right\}du. \tag{30}$$

Using relation (30) and properties of the function $\psi \in \mathfrak{M}$, we get

$$\int\limits_{1/2}^{\infty} |u-1|\left|d\tau'(u)\right| \le \int\limits_{1/2}^{\infty} u|d\tau'(u)|$$

$$\le \frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\left(1-e^{-u}\right)\delta^2\psi''(\delta u)du + \frac{2\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}\left|\psi'(\delta u)\right|du$$

$$+ \frac{1}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}\psi(\delta u)du. \tag{31}$$

Since $1-e^{-u} \le 1$ for $u \ge 0$, $ue^{-u} \le K$, and $\psi(\delta u) \le \psi(\delta/2)$ for $u \in [1/2;\infty)$, $\delta \ge 2$, relation (31) yields

$$\int\limits_{1/2}^{\infty} |u-1|\left|d\tau'(u)\right| \le \frac{\delta^2}{\psi(\delta)}\int\limits_{1/2}^{\infty} u\psi''(\delta u)du + \frac{2K\delta}{\psi(\delta)}\int\limits_{1/2}^{\infty}\left|\psi'(\delta u)\right|du + \frac{\psi\left(\dfrac{\delta}{2}\right)}{\psi(\delta)}\int\limits_{1/2}^{\infty} ue^{-u}du. \tag{32}$$

By virtue of Proposition 3, we obtain the following relation for the continuous function $\psi(\delta u) \in \mathfrak{M}_0$, $u \geq 1/2$, $\delta \geq 2$:

$$\frac{\delta}{\psi(\delta)} \int\limits_{1/2}^{\infty} \left|\psi'(\delta u)\right| du = -\frac{1}{\psi(\delta)} \int\limits_{1/2}^{\infty} d\psi(\delta u) \leq K. \tag{33}$$

Further, we show that, for any function $\psi \in \mathfrak{M}$, one has

$$\lim_{u\to\infty} u\psi'(u) = 0. \tag{34}$$

Indeed, since the function $|\psi'(u)|$ is decreasing for $u \geq 1$, we get

$$\frac{1}{2}\lim_{u\to\infty} u|\psi'(u)| = \lim_{\delta\to\infty} \frac{\delta}{2}|\psi'(\delta)| = \lim_{\delta\to\infty}\left(\delta - \frac{\delta}{2}\right)|\psi'(\delta)|$$

$$\leq \lim_{\delta\to\infty} \int\limits_{\delta/2}^{\delta} |\psi'(u)|du \leq -\lim_{\delta\to\infty} \int\limits_{\delta/2}^{\infty} \psi'(u)du = \lim_{\delta\to\infty} \psi\left(\frac{\delta}{2}\right) = 0.$$

Let us estimate the first integral on the right-hand side of (32). Taking into account relations (33) and (34) and Propositions 2 and 3, we obtain

$$\frac{\delta^2}{\psi(\delta)} \int\limits_{1/2}^{\infty} u\psi''(\delta u)du = \frac{\delta}{\psi(\delta)} \int\limits_{1/2}^{\infty} u d\psi'(\delta u)$$

$$= \frac{\delta}{\psi(\delta)} \lim_{u\to\infty} u\psi'(\delta u) + \frac{\frac{\delta}{2}\left|\psi'\left(\frac{\delta}{2}\right)\right|}{\psi(\delta)} + \frac{\delta}{\psi(\delta)} \int\limits_{1/2}^{\infty} \left|\psi'(\delta u)\right| du \leq K_1. \tag{35}$$

Combining relations (32), (33), and (35), we get

$$\int\limits_{1/2}^{\infty} |u-1|\left|d\tau'(u)\right| = O(1). \tag{36}$$

To estimate the first integral in (11), we divide the segment $[0;\infty)$ into three parts: $[0;1/\delta]$, $[1/\delta;1]$, and $[1,\infty)$. Using relations (5) and (12), we obtain

$$\int\limits_{0}^{1/\delta} \frac{\tau(u)}{u} du = \frac{\psi(1)}{\psi(\delta)} \int\limits_{0}^{1/\delta} \left(1 - e^{-u}\right) \frac{du}{u} \leq \frac{\psi(1)}{\psi(\delta)} \int\limits_{0}^{1/\delta} u \frac{du}{u} = \frac{\psi(1)}{\delta\psi(\delta)}. \tag{37}$$

Using relations (5), (17), and (19) and estimates (23) and (24), we get

$$\left|\int_{1/\delta}^{1}\frac{\tau(u)}{u}du-\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\psi(\delta u)du\right|\le\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\frac{|\overline{\mu}(u)|}{u}\psi(\delta u)du$$

$$\le\frac{1}{2\psi(\delta)}\left(\int_{1/\delta}^{b/\delta}+\int_{b/\delta}^{1}\right)u\psi(\delta u)du=O\left(1+\frac{1}{\delta\psi(\delta)}\right).$$

Hence,

$$\int_{1/\delta}^{1}\frac{\tau(u)}{u}du=\frac{1}{\delta\psi(\delta)}\int_{1}^{\delta}\psi(u)du+O\left(1+\frac{1}{\delta\psi(\delta)}\right),\quad\delta\to\infty. \tag{38}$$

Taking into account that the function $\psi(u)$ decreases for $u\ge 1$, we obtain

$$\left|\int_{1}^{\infty}\frac{\tau(u)}{u}du-\frac{1}{\psi(\delta)}\int_{\delta}^{\infty}\frac{\psi(u)}{u}du\right|=\frac{1}{\psi(\delta)}\int_{1}^{\infty}\frac{e^{-u}}{u}\psi(\delta u)du\le\int_{1}^{\infty}\frac{e^{-u}}{u}du\le K. \tag{39}$$

It follows from relations (37)–(39) that

$$\int_{0}^{\infty}\frac{|\tau(u)|}{u}du=\frac{1}{\delta\psi(\delta)}\int_{1}^{\delta}\psi(u)du+\frac{1}{\psi(\delta)}\int_{\delta}^{\infty}\frac{\psi(u)}{u}du+O\left(1+\frac{1}{\delta\psi(\delta)}\right). \tag{40}$$

Let us estimate the second integral in (11). Using relation (5), we get

$$\tau(1-u)=\begin{cases}\left(1-e^{-(1-u)}\right)\dfrac{\psi(1)}{\psi(\delta)}, & 1-\dfrac{1}{\delta}\le u\le 1,\\[2ex] \left(1-e^{-(1-u)}\right)\dfrac{\psi(\delta(1-u))}{\psi(\delta)}, & u\le 1-\dfrac{1}{\delta},\end{cases} \tag{41}$$

$$\tau(1+u)=\begin{cases}\left(1-e^{-(1+u)}\right)\dfrac{\psi(1)}{\psi(\delta)}, & -1\le u\le\dfrac{1}{\delta}-1,\\[2ex] \left(1-e^{-(1+u)}\right)\dfrac{\psi(\delta(1+u))}{\psi(\delta)}, & u\ge\dfrac{1}{\delta}-1.\end{cases} \tag{42}$$

We represent the second integral in (11) as a sum of two integrals:

$$\int_{0}^{1}\frac{|\tau(1-u)-\tau(1+u)|}{u}du=\int_{0}^{1-1/\delta}\frac{|\tau(1-u)-\tau(1+u)|}{u}du+\int_{1-1/\delta}^{1}\frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{43}$$

First, we estimate the first term on the right-hand side of (43). To this end, we add and subtract the term

$$e^{-(1-u)} - e^{-(1+u)}$$

under the modulus sign in the integrand. As a result, we get

$$\int\limits_0^{1-1/\delta} \frac{|\tau(1-u)-\tau(1+u)|}{u}du$$

$$\leq \int\limits_0^{1-1/\delta} \frac{\left|e^{-(1-u)}-e^{-(1+u)}\right|}{u}du + \int\limits_0^{1-1/\delta} \frac{\left|\tau(1-u)-\tau(1+u)+e^{-(1-u)}-e^{-(1+u)}\right|}{u}du. \qquad (44)$$

For the first integral on the right-hand side of (44), we obtain the obvious estimate

$$\int\limits_0^{1-1/\delta} \left|e^{-1+u}-e^{-1-u}\right| \frac{du}{u} = O(1). \qquad (45)$$

By virtue of (41) and (42), we obtain the following relations for $u \in [0, 1-1/\delta]$:

$$e^{-(1-u)} = 1 - \frac{\psi(\delta)}{\psi(\delta(1-u))}\tau(1-u), \qquad e^{-(1+u)} = 1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\tau(1+u).$$

Then

$$\int\limits_0^{1-1/\delta} \frac{\left|\tau(1-u)-\tau(1+u)+e^{-(1-u)}-e^{-(1+u)}\right|}{u}du$$

$$\leq \int\limits_0^{1-1/\delta} \left|\tau(1-u)\right| \left|1-\frac{\psi(\delta)}{\psi(\delta(1-u))}\right| \frac{du}{u} + \int\limits_0^{1-1/\delta} \left|\tau(1+u)\right| \left|1-\frac{\psi(\delta)}{\psi(\delta(1+u))}\right| \frac{du}{u}. \qquad (46)$$

Since a function $\tau(\cdot)$ of the form (5) belongs to the set $\mathcal{E}_1$, Proposition 1 is true. According to this proposition,

$$\int\limits_0^{1-1/\delta} \left|\tau(1-u)\right| \left|1-\frac{\psi(\delta)}{\psi(\delta(1-u))}\right| \frac{du}{u} + \int\limits_0^{1-1/\delta} \left|\tau(1+u)\right| \left|1-\frac{\psi(\delta)}{\psi(\delta(1+u))}\right| \frac{du}{u}$$

$$= H(\tau)O\left(\int\limits_0^{1-1/\delta} \frac{|\psi(\delta(1-u))-\psi(\delta)|}{u\psi(\delta(1-u))}du + \int\limits_0^{1-1/\delta} \frac{|\psi(\delta(1+u))-\psi(\delta)|}{u\psi(\delta(1+u))}du \right). \qquad (47)$$

We show that, as $\delta \to \infty$,

$$I_{1,\delta} := \int_0^{1-1/\delta} \frac{|\psi(\delta(1-u)) - \psi(\delta)|}{u\psi(\delta(1-u))} du = O(1), \tag{48}$$

$$I_{2,\delta} := \int_0^{1-1/\delta} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du = O(1), \tag{49}$$

where $O(1)$ is a quantity uniformly bounded in δ.

Indeed, the function

$$\frac{1 - \psi(\delta)/\psi(\delta(1-u))}{u}$$

is bounded for all $u \in [\delta, 1-1/\delta]$, $0 < \delta < 1-1/\delta$, and, moreover, with regard for Propositions 2, for $\psi \in \mathfrak{M}_0$ we have

$$\lim_{u\to 0} \frac{1 - \psi(\delta)/\psi(\delta(1-u))}{u} = \frac{\delta\,|\psi'(\delta)|}{\psi(\delta)} \le K.$$

Thus, $I_{1,\delta} = O(1)$, $\delta \to \infty$. Passing to the estimation of the integral $I_{2,\delta}$, note that

$$I_{2,\delta} < \frac{1}{\psi(2\delta - 1)} \int_0^{1-1/\delta} \frac{\psi(\delta) - \psi\,(\delta\,(1+u))}{u} du.$$

Performing the change of variables $v = \delta(1+u)$, we get

$$I_{2,\delta} < \frac{1}{\psi(2\delta-1)} \int_\delta^{2\delta-1} \frac{\psi(\delta) - \psi\,(v)}{v-\delta} dv < \frac{1}{\psi(2\delta-1)} \int_\delta^{2\delta} \frac{\psi(\delta) - \psi\,(v)}{v-\delta} dv.$$

Applying Lemma 5.5 from [3, p. 97] to the right-hand side of the last inequality, taking into account that $\psi(2\delta - 1) \ge \psi(2\delta)$, $\delta \ge 1$, and using Proposition 3, we obtain

$$I_{2,\delta} < \frac{K_1\psi(\delta)}{\psi(2\delta-1)} \le \frac{K_1\psi(\delta)}{\psi(2\delta)} \le K_2.$$

Combining relations (46)–(49), we get

$$\int_0^{1-1/\delta} \frac{\left|\tau(1-u) - \tau(1+u) + e^{-(1-u)} - e^{-(1+u)}\right|}{u} du = H(\tau)O(1), \quad \delta \to \infty.$$

According to (5), (29), and (36), quantities $H(\tau)$ of the form (7) satisfy the estimate

$$H(\tau) = O\left(1 + \frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty. \tag{50}$$

Thus,

$$\int\limits_0^{1-1/\delta} \frac{\left|\tau(1-u) - \tau(1+u) + e^{-(1-u)} - e^{-(1+u)}\right|}{u} du = O\left(1 + \frac{1}{\delta\psi(\delta)}\right) \quad \text{as} \quad \delta \to \infty. \tag{51}$$

Comparing (44), (45), and (51), we obtain

$$\int\limits_0^{1-1/\delta} \frac{|\tau(1-u) - \tau(1+u)|}{u} du = O\left(1 + \frac{1}{\delta\psi(\delta)}\right). \tag{52}$$

Let us estimate the second term on the right-hand side of (43). We have

$$\int\limits_{1-1/\delta}^{1} \frac{|\tau(1-u) - \tau(1+u)|}{u} du$$

$$= \int\limits_{1-1/\delta}^{1} \frac{\left|e^{-(1-u)} - e^{-(1+u)}\right|}{u} du + O\left(\int\limits_{1-1/\delta}^{1} \frac{\left|\tau(1-u) - \tau(1+u) + e^{-(1-u)} - e^{-(1+u)}\right|}{u} du\right). \tag{53}$$

Using relations (41) and (42), we obtain the following equalities for $u \in [1 - 1/\delta; 1]$:

$$e^{-(1-u)} = 1 - \frac{\psi(\delta)}{\psi(1)}\tau(1-u), \qquad e^{-(1+u)} = 1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\tau(1+u).$$

Using these equalities and Proposition 1, we get

$$\int\limits_{1-1/\delta}^{1} \frac{\left|\tau(1-u) - \tau(1+u) + e^{-(1-u)} - e^{-(1+u)}\right|}{u} du$$

$$= \int\limits_{1-1/\delta}^{1} \left|\tau(1-u)\left(1 - \frac{\psi(\delta)}{\psi(1)}\right) - \tau(1+u)\left(1 - \frac{\psi(\delta)}{\psi(\delta(1+u))}\right)\right| \frac{du}{u}$$

$$= H(\tau) O\left(\int\limits_{1-1/\delta}^{1} \frac{|\psi(1) - \psi(\delta)|}{u\psi(1)} du + \int\limits_{1-1/\delta}^{1} \frac{|\psi(\delta(1+u)) - \psi(\delta)|}{u\psi(\delta(1+u))} du\right). \tag{54}$$

Further, we obtain

$$\int\limits_{1-1/\delta}^{1} \frac{|\psi(1)-\psi(\delta)|}{u\psi(1)}du = \left(1-\frac{\psi(\delta)}{\psi(1)}\right)\ln\frac{1}{1-1/\delta} = O(1). \tag{55}$$

Repeating the arguments used in the derivation of estimate (49), we show that

$$\int\limits_{1-1/\delta}^{1} \frac{|\psi(\delta(1+u))-\psi(\delta)|}{u\psi(\delta(1+u))}du = O(1) \quad \text{as} \quad \delta\to\infty. \tag{56}$$

Combining (53)–(56) and using relation (50) and the fact that

$$\int\limits_{1-1/\delta}^{1} \frac{\left|e^{-(1-u)}-e^{-(1+u)}\right|}{u}du = O(1),$$

we obtain

$$\int\limits_{1-1/\delta}^{1} \frac{\left|\tau(1-u)-\tau(1+u)\right|}{u}du = O\left(1+\frac{1}{\delta\psi(\delta)}\right), \quad \delta\to\infty. \tag{57}$$

Using equality (43) and estimates (52) and (57), we get

$$\int\limits_{0}^{1} |\tau(1-u)-\tau(1+u)|\,\frac{du}{u} = O\left(1+\frac{1}{\delta\psi(\delta)}\right), \quad \delta\to\infty. \tag{58}$$

Thus, by virtue of Theorem 1 in [9], an integral $A(\tau)$ of the form (9) is convergent.
Lemma 1 is proved.

3. Asymptotic Equalities for Upper Bounds of Deviations of Poisson Integrals from Functions of the Classes $C^{\psi}_{\beta,\infty}$ in the Uniform Metric

The following statement is true:

Theorem 1. *Let* $\psi\in\mathfrak{M}'_0$ *and let the function* $g(u)=u\psi(u)$ *be convex upward or downward on* $[b,\infty)$, $b\ge 1$. *Then the following equality holds as* $\delta\to\infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty};P_\delta\right)_C = \psi(\delta)A(\tau,\delta)+O\left(\frac{1}{\delta}\right), \tag{59}$$

where $A(\tau)$ *is defined by (9) and satisfies the estimate*

$$A(\tau,\delta)=\frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\delta\psi(\delta)}\int\limits_{1}^{\delta}\psi(u)du+\frac{1}{\psi(\delta)}\int\limits_{\delta}^{\infty}\frac{\psi(u)}{u}du\right)+O\left(1+\frac{1}{\delta\psi(\delta)}\right). \tag{60}$$

Proof. It is shown in Lemma 1 that the Fourier transform of the function $\tau(u)$ defined by (5) is summable on the entire number axis, i.e., an integral $A(\tau)$ of the form (9) is convergent. Repeating the arguments used in [4, p. 183], we establish that the following equality holds for any function $f \in C^{\psi}_{\beta,\infty}$ at every point $x \in R$:

$$f(x) - P_\delta(f; x) = \psi(\delta) \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right) \hat{\tau}_\delta(t)dt, \quad \delta > 0. \tag{61}$$

Using (2) and (61) and taking into account that the classes $C^{\psi}_{\beta,\infty}$ are invariant under translation of the argument (see [3, p. 109]), we obtain

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \sup_{f \in C^{\psi}_{\beta,\infty}} \left| \psi(\delta) \int\limits_{-\infty}^{+\infty} f^{\psi}_{\beta}\left(\frac{t}{\delta}\right) \hat{\tau}(t)dt \right|.$$

Hence,

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C \leq \frac{\psi(\delta)}{\pi} \int\limits_{-\infty}^{+\infty} \left| \int\limits_{0}^{\infty} \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt. \tag{62}$$

On the other hand, for any function $\varphi_0 \in L_1$ such that

$$\int\limits_{-\pi}^{\pi} \varphi_0(t)dt = 0 \qquad \text{and} \qquad \operatorname*{ess\,sup}_{t} |\varphi_0(t)| \leq 1,$$

the class $C^{\psi}_{\beta,\infty}$ contains a function $f(x) = f(\varphi_0; x)$ for which $f^{\psi}_{\beta}(x) = \varphi_0(x)$. Therefore, the class $C^{\psi}_{\beta,\infty}$ contains a function $\hat{f}(t)$ such that

$$\hat{f}^{\psi}_{\beta}(t) = \operatorname{sign} \int\limits_{0}^{\infty} \tau(u) \cos\left(u\delta t + \frac{\beta\pi}{2}\right) du, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{63}$$

Further, since

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C \geq \frac{\psi(\delta)}{\pi} \left| \int\limits_{-\infty}^{+\infty} \hat{f}^{\psi}_{\beta}\left(\frac{t}{\delta}\right) \int\limits_{0}^{\infty} \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du dt \right|, \tag{64}$$

using (63) we obtain

$$\frac{\psi(\delta)}{\pi}\left|\int\limits_{-\infty}^{+\infty}\hat{f}_{\beta}^{\psi}\left(\frac{t}{\delta}\right)\int\limits_{0}^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du dt\right|$$

$$\geq \delta\psi(\delta)\left|\int\limits_{-\pi/2}^{\pi/2}\operatorname{sign}\hat{\tau}(t\delta)\hat{\tau}(t\delta)dt\right|-\psi(\delta)\int\limits_{|t|\geq(\delta\pi/2)}|\hat{\tau}_{\delta}(t)|\,dt$$

$$=\psi(\delta)\int\limits_{-\infty}^{+\infty}|\hat{\tau}_{\delta}(t)|\,dt+\gamma(\delta), \tag{65}$$

where $\gamma(\delta)\leq 0$ and

$$|\gamma(\delta)|=O\left(\psi(\delta)\int\limits_{|t|\geq(\delta\pi/2)}|\hat{\tau}_{\delta}(t)|\,dt\right).$$

Combining relations (62), (64), and (65), we get

$$\mathcal{E}\left(C_{\beta,\infty}^{\psi};P_{\delta}\right)_{C}=\psi(\delta)A(\tau)+O\left(\psi(\delta)\int\limits_{|t|\geq(\delta\pi/2)}|\hat{\tau}_{\delta}(t)|\,dt\right)\quad\text{as}\quad\delta\to\infty. \tag{66}$$

Moreover, using inequalities (2.14) and (2.15) of [9, p. 25] and relations (29), (36), (40), and (58) of the present paper, we obtain equality (60).

Let us estimate the remainder on the right-hand side of (66). To this end, we rewrite the transform $\hat{\tau}_{\delta}(t)$ defined by (4) as follows:

$$\hat{\tau}(t)=\frac{1}{\pi}\left(\int\limits_{0}^{1/\delta}+\int\limits_{1/\delta}^{\infty}\right)\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{67}$$

Integrating both integrals in (67) twice by parts and taking into account that $\tau(0)=0$ and

$$\lim_{u\to\infty}\tau(u)=\lim_{u\to\infty}\tau'(u)=0,$$

we obtain

$$\int\limits_{0}^{1/\delta}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=\frac{1}{t}\tau\left(\frac{1}{\delta}\right)\sin\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)+\frac{1}{t^2}\tau'\left(\frac{1}{\delta}\right)\cos\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\tau'(0)\cos\frac{\beta\pi}{2}-\frac{1}{t^2}\int\limits_{0}^{1/\delta}\tau''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du, \tag{68}$$

$$\int_{1/\delta}^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=-\frac{1}{t}\tau\left(\frac{1}{\delta}\right)\sin\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)-\frac{1}{t^2}\tau'\left(\frac{1}{\delta}\right)\cos\left(\frac{t}{\delta}+\frac{\beta\pi}{2}\right)$$

$$-\frac{1}{t^2}\int_{1/\delta}^{\infty}\tau''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{69}$$

Combining relations (68) and (69), we get

$$\int_{0}^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du$$

$$=-\frac{1}{t^2}\tau'(0)\cos\frac{\beta\pi}{2}-\frac{1}{t^2}\int_{0}^{1/\delta}\tau''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du-\frac{1}{t^2}\int_{1/\delta}^{\infty}\tau''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du.$$

Hence,

$$\left|\int_{0}^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{K_1}{t^2\psi(\delta)}+\frac{1}{t^2}\left(\int_{0}^{1/\delta}+\int_{1/\delta}^{1}+\int_{1}^{\infty}\right)\left|\tau''(u)\right|du. \tag{70}$$

Let us estimate the integrals on the right-hand side of (70). Taking into account that $\tau''(u)<0$ for $u\in[0;1/\delta]$ and using inequality (12), we get

$$\int_{0}^{1/\delta}|\tau''(u)|du=-\int_{0}^{1/\delta}\tau''(u)du=\frac{\psi(1)}{\psi(\delta)}\,e^{-u}\big|_{0}^{1/\delta}=O\left(\frac{1}{\delta\psi(\delta)}\right). \tag{71}$$

Taking (5), (14), and (15) into account, we estimate the second integral on the right-hand side of (70):

$$\int_{1/\delta}^{1}|\tau''(u)|du\le\int_{1/\delta}^{1}|\tau_1''(u)|du+\int_{1/\delta}^{1}|\tau_2''(u)|du. \tag{72}$$

With regard for (19) and (20), we get

$$\int_{1/\delta}^{1}|\tau_1''(u)|du\le\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\frac{u^2}{2}\delta^2\psi''(\delta u)du+\frac{2}{\psi(\delta)}\int_{1/\delta}^{1}u\delta\left|\psi'(\delta u)\right|du+\frac{1}{\psi(\delta)}\int_{1/\delta}^{1}\psi(\delta u)du. \tag{73}$$

Integrating the first integral on the right-hand side of the last inequality by parts and taking into account the conditions of Proposition 2, we obtain the following inequality for the function $\psi(u) \in \mathfrak{M}_0$, $u \geq 1$:

$$\frac{\delta^2}{2\psi(\delta)} \int\limits_{1/\delta}^{1} u^2 \psi''(\delta u) du \leq K + \frac{|\psi'(1)|}{2\delta\psi(\delta)} + \frac{\delta}{\psi(\delta)} \int\limits_{1/\delta}^{1} u \left|\psi'(\delta u)\right| du. \tag{74}$$

Since

$$\int\limits_{1/\delta}^{1} \psi(\delta u) du = \frac{1}{\delta} \int\limits_{1}^{\delta} \psi(u) du \leq \psi(1)\left(1 - \frac{1}{\delta}\right),$$

using Proposition 2 we establish that

$$\frac{\delta}{\psi(\delta)} \int\limits_{1/\delta}^{1} u \left|\psi'(\delta u)\right| du \leq \frac{K_1}{\psi(\delta)} \int\limits_{1/\delta}^{1} \psi(\delta u) du \leq \frac{K_2}{\psi(\delta)}. \tag{75}$$

Combining relations (73)–(75), we get

$$\int\limits_{1/\delta}^{1} |\tau_1''(u)| du \leq K + \frac{K_1}{\delta\psi(\delta)} + \frac{K_2}{\psi(\delta)}. \tag{76}$$

To estimate the second integral on the right-hand side of inequality (72), we represent it in the form

$$\int\limits_{1/\delta}^{1} |\tau_2''(u)| du = \left(\int\limits_{1/\delta}^{b/\delta} + \int\limits_{b/\delta}^{1} \right) |\tau_2''(u)| du, \quad \delta > b. \tag{77}$$

With regard for relation (26), we get

$$\int\limits_{1/\delta}^{b/\delta} |\tau_2''(u)| du \leq \frac{\delta^2}{\psi(\delta)} \int\limits_{1/\delta}^{b/\delta} u\psi''(\delta u) du + \frac{2\delta}{\psi(\delta)} \int\limits_{1/\delta}^{b/\delta} \left|\psi'(\delta u)\right| du$$

$$= \frac{b\psi'(b) - \psi'(1)}{\psi(\delta)} - \frac{3\delta}{\psi(\delta)} \int\limits_{1/\delta}^{b/\delta} \psi'(\delta u) du = O\left(\frac{1}{\psi(\delta)}\right). \tag{78}$$

Since, according to the conditions of the theorem, the function $g(u) = u\psi(u)$ is convex on $[b; \infty)$, using (15) we obtain the following estimate:

$$\int\limits_{b/\delta}^{1} |\tau_2''(u)| du = \left| \int\limits_{b/\delta}^{1} \tau_2''(u) du \right| = O\left(\frac{1}{\psi(\delta)}\right) \quad \text{as} \quad \delta \to \infty. \tag{79}$$

It follows from (72) and (76)–(79) that

$$\int_{1/\delta}^{1} |\tau''(u)|du = O\left(\frac{1}{\psi(\delta)}\right), \quad \delta \to \infty. \tag{80}$$

Consider the integral on the right-hand side of (70) on the interval $[1, \infty)$. Using (30), we obtain

$$\int_{1}^{\infty} |\tau''(u)|du \leq \frac{\delta^2}{\psi(\delta)} \int_{1}^{\infty} \left(1 - e^{-u}\right) \psi''(\delta u)du + \frac{2\delta}{\psi(\delta)} \int_{1}^{\infty} e^{-u} \left|\psi'(\delta u)\right| du + \frac{1}{\psi(\delta)} \int_{1}^{\infty} e^{-u} \psi(\delta u)du.$$

Using the inequalities $1 - e^{-u} \leq u$ and $e^{-u} \leq 1$ for $u \geq 0$ and $\psi(\delta u) \leq \psi(\delta)$ for $u \geq 1,$ Propositions 2 and 3, and relation (34), we get

$$\int_{1}^{\infty} |\tau''(u)|du = O(1), \quad \delta \to \infty. \tag{81}$$

Relations (71), (80), and (81) yield

$$\int_{0}^{\infty} |\tau''(u)|du = O\left(\frac{1}{\psi(\delta)}\right).$$

Taking into account the last estimate and inequality (70), we obtain

$$\int_{|t| \geq \delta\pi/2} \left| \int_{0}^{\infty} \tau(u) \cos\left(ut + \frac{\beta\pi}{2}\right) du \right| dt = O\left(\frac{1}{\delta\psi(\delta)}\right), \quad \delta \to \infty. \tag{82}$$

Equality (59) follows from relations (82) and (66).

Theorem 1 is proved.

Note that, for the classes $C^{\psi}_{\beta,\infty}$ of periodic functions, an analogous theorem was established in [9, p. 31] in the case of

$$\psi(u) = \frac{1}{u^r}, \quad 0 < r < 1, \quad u \geq 1.$$

Corollary 1. *Suppose that the conditions of Theorem 1 are satisfied,*

$$\sin \frac{\beta\pi}{2} \neq 0,$$

and

$$\lim_{t \to \infty} \alpha(t) = \infty,$$

where $\alpha(t)$ *is defined by (8). Then the following asymptotic equality is true:*

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \int\limits_{\delta}^{\infty} \frac{\psi(u)}{u} du + O\left(\psi(\delta)\right) \quad as \quad \delta \to \infty. \tag{83}$$

Proof. To verify equality (83), we first note that, for $\varepsilon_0 \in (0,1)$, the function $u^{\varepsilon_0}\psi(u)$ increases beginning with a certain number $u_0 \geq 1$. Indeed,

$$\left(u^{\varepsilon_0}\psi(u)\right)' = \varepsilon_0 u^{\varepsilon_0 - 1}\psi(u) - u^{\varepsilon_0}|\psi'(u)| = u^{\varepsilon_0}|\psi'(u)|\left(\varepsilon_0\alpha(u) - 1\right).$$

Since

$$\lim_{u\to\infty} \alpha(u) = \infty,$$

there exists $u_0 = u_0(\varepsilon_0)$ such that $(u^{\varepsilon_0}\psi(u))' > 0$ for $u > u_0$. Then the following relation holds for any $\varepsilon \in (\varepsilon_0, 1)$ and sufficiently large δ:

$$\frac{1}{\delta\psi(\delta)} \int\limits_1^{\delta} \psi(u)du = \frac{1}{\delta\psi(\delta)} \int\limits_1^{\delta} \frac{u^{\varepsilon}\psi(u)}{u^{\varepsilon}} du \leq \frac{\delta^{\varepsilon}\psi(\delta)}{\delta\psi(\delta)} \int\limits_1^{\delta} \frac{du}{u^{\varepsilon}} = O(1). \tag{84}$$

Since $\psi \in \mathfrak{M}'_0$, using the l'Hospital rule and the fact that

$$\lim_{u\to\infty} \alpha(u) = \infty$$

we get

$$\lim_{x\to\infty} \frac{\displaystyle\int_x^{\infty} \frac{\psi(u)}{u} du}{\psi(x)} = \lim_{x\to\infty} \frac{\psi(x)}{x|\psi'(x)|} = \infty. \tag{85}$$

Thus,

$$\psi(\delta) = o\left(\int\limits_{\delta}^{\infty} \frac{\psi(u)}{u} du\right) \quad \text{as} \quad \delta \to \infty. \tag{86}$$

Combining (84) and (86) with (59) and (60), we obtain (83).

Examples of functions that satisfy the conditions of Corollary 1 are functions of the form

$$\psi(u) = \frac{1}{\ln^{\alpha}(u+K)},$$

where $\alpha > 1$ and $K > 0$.

Corollary 2. *Suppose that* $\psi \in \mathfrak{M}_0$,

$$\sin \frac{\beta\pi}{2} \neq 0,$$

the function $u\psi(u)$ *is convex upward or downward on* $[b, \infty)$, $b \geq 1$, *and*

$$\lim_{u \to \infty} u\psi(u) = \infty, \tag{87}$$

$$\lim_{\delta \to \infty} \frac{1}{\delta\psi(\delta)} \int_1^\delta \psi(u)du = \infty. \tag{88}$$

Then the following asymptotic equality is true:

$$\mathcal{E}\left(C^\psi_{\beta,\infty}; P_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta} \int_1^\delta \psi(u)du + O\left(\psi(\delta)\right) \quad as \quad \delta \to \infty. \tag{89}$$

Proof. If the function ψ satisfies conditions (87) and (88), then, using the l'Hospital rule, we obtain

$$\frac{1}{1 - \lim\limits_{x\to\infty} \alpha(x)} = \lim_{x\to\infty} \frac{\psi(x)}{\psi(x) + x\psi'(x)} = \lim_{x\to\infty} \frac{\int_1^x \psi(u)du}{x\psi(x)} = \infty.$$

Hence,

$$\lim_{x\to\infty} \alpha(x) = 1. \tag{90}$$

It follows from (85) and (90) that

$$\int_\delta^\infty \frac{\psi(u)}{u} du = O\left(\psi(\delta)\right).$$

Using the last estimate and relations (59), (60), (87), and (88), we get (89).

Examples of functions that satisfy the conditions of Corollary 2 are functions of the form

$$\psi(u) = \frac{1}{u}\ln^\alpha(u + K),$$

where $K > 0$ and $\alpha > 0$.

Corollary 3. *Suppose that* $\psi \in \mathfrak{M}_0$,

$$\sin \frac{\beta\pi}{2} \neq 0,$$

the function $u\psi(u)$ *is convex downward on* $[b,\infty)$, $b \geq 1$, *and*

$$\lim_{u\to\infty} u\psi(u) = K < \infty, \tag{91}$$

$$\lim_{\delta\to\infty} \int_1^\delta \psi(u)du = \infty. \tag{92}$$

Then the following asymptotic equality is true:

$$\mathcal{E}\left(C^{\psi}_{\beta,\infty}; P_\delta\right)_C = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta}\int_1^\delta \psi(u)du + O\left(\frac{1}{\delta}\right) \quad as \quad \delta\to\infty. \tag{93}$$

Proof. Taking into account that, under the conditions of Corollary 3, the function $u\psi(u)$ is decreasing for $u \geq b \geq 1$, for sufficiently large δ $(\delta > b)$ we get

$$\frac{1}{\psi(\delta)}\int_\delta^\infty \frac{\psi(u)}{u}du = \frac{1}{\psi(\delta)}\int_\delta^\infty \frac{u\psi(u)}{u^2}du \leq \delta \int_\delta^\infty \frac{du}{u^2} = O(1).$$

We obtain equality (93) by substituting the last expression into (60) and taking relations (59), (91), and (92) into account.

Examples of functions for which Corollary 3 is true are functions of the form

$$\psi(u) = \frac{1}{u}(K + e^{-u}) \qquad \text{and} \qquad \psi(u) = \frac{1}{u}\ln^\alpha(u+K),$$

where $K > 0$ and $-1 \leq \alpha \leq 0$.

If

$$\psi(u) = \frac{1}{u}, \quad u \geq 1, \quad \beta \in R,$$

then relation (93) yields equality (3) (see [9, p. 31]). For

$$\psi(u) = \frac{1}{u^r}, \quad u \geq 1, \quad \beta = r = 1,$$

relation (93) yields the following asymptotic equality:

$$\mathcal{E}\left(W^1_\infty; P_\delta\right)_C = \frac{2}{\pi}\frac{\ln\delta}{\delta} + O\left(\frac{1}{\delta}\right) \quad \text{as} \quad \delta\to\infty.$$

This estimate for upper bounds of approximations by Poisson integrals on the Sobolev classes W^1_∞ was obtained by Natanson in [7].

Note that, under the conditions of Corollaries 1–3, equalities (83), (89), and (93) give a solution of the Kolmogorov–Nikol'skii problem for Poisson integrals on the classes $C^{\psi}_{\beta,\infty}$ in the uniform metric in the case where the functions ψ decrease slowly to zero, i.e., in the case where

$$\int_{1}^{\infty} \psi(u)du = \infty.$$

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REFERENCES

1. A. I. Stepanets, *Classes of Periodic Functions and Approximation of Their Elements by Fourier Sums* [in Russian], Preprint No. 83.10, Institute of Mathematics, Academy of Sciences of Ukr. SSR, Kiev (1983).
2. A. I. Stepanets, "Deviations of Fourier sums on classes of infinitely differentiable functions," *Ukr. Mat. Zh.*, **36**, No. 6, 750–758 (1984).
3. A. I. Stepanets, *Classification and Approximation of Periodic Functions* [in Russian], Naukova Dumka, Kiev (1987).
4. A. I. Stepanets, *Methods of Approximation Theory* [in Russian], Vol. 1, Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (2002).
5. B. Sz.-Nagy, "Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen. I," *Berichte Akad. Wiss.*, **90**, 103–134 (1938).
6. A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge University, Cambridge (1959).
7. I. P. Natanson, "On the order of approximation of a continuous 2π-periodic function by its Poisson integral," *Dokl. Akad. Nauk SSSR*, **72**, No. 1, 11–14 (1950).
8. A. F. Timan, "Sharp estimate for a remainder in the approximation of periodic differentiable functions by Poisson integrals," *Dokl. Akad. Nauk SSSR*, **74**, No. 1, 17–20 (1950).
9. L. I. Bausov, "Linear methods for summation of Fourier series with given rectangular matrices. I," *Izv. Vyssh. Uchebn. Zaved., Ser. Mat.*, **46**, No. 3, 15–31 (1965).
10. K. M. Zhyhallo and Yu. I. Kharkevych, "Complete asymptotics of the deviation of a class of differentiable functions from the set of their harmonic Poisson integrals," *Ukr. Mat. Zh.*, **54**, No. 1, 43–52 (2002).
11. K. M. Zhyhallo and Yu. I. Kharkevych, "Approximation of conjugate differentiable functions by their Abel–Poisson integrals," *Ukr. Mat. Zh.*, **61**, No. 1, 73–82 (2009).